Mathematical Induction - Problems With Solutions

Several problems with detailed solutions on mathematical induction are presented.

The principle of mathematical induction is used to prove that a given proposition (formula, equality, inequality…) is true for all positive integer numbers greater than or equal to some integer N.

Let us denote the proposition in question by P (n), where n is a positive integer.

The proof involves two steps:

Step 1: We first establish that the proposition P (n) is true for the lowest possible value of the positive integer n.

Step 2: We assume that P (k) is true and establish that P (k+1) is also true

Problem 1

Use mathematical induction to prove that

$1 + 2 + 3 + ... + n = n (n + 1) / 2$

for all positive integers n.

Solution to Problem 1:

Let the statement P (n) be

$1 + 2 + 3 + ... + n = n (n + 1) / 2$

STEP 1: We first show that p (1) is true.

Left Side = 1

Right Side = 1 (1 + 1) / 2 = 1

Both sides of the statement are equal hence p (1) is true.

STEP 2: We now assume that p (k) is true

$1 + 2 + 3 + ... + k = k (k + 1) / 2$

and show that p (k + 1) is true by adding k + 1 to both sides of the above statement

$1 + 2 + 3 + ... + k + (k + 1) = k (k + 1) / 2 + (k + 1)$

$= (k + 1)(k / 2 + 1)$

$= (k + 1)(k + 2) / 2$

The last statement may be written as

$1 + 2 + 3 + ... + k + (k + 1) = (k + 1)(k + 2) / 2$

Which is the statement p(k + 1).

Problem 2

Prove that

$$\mathbf{1^2 + 2^2 + 3^2 + ... + n^2 = n (n + 1) (2n + 1)/ 6}$$

For all positive integers n.

Solution to Problem 2:

Statement P (n) is defined by

$1^2 + 2^2 + 3^2 + ... + n^2 = n (n + 1) (2n + 1)/ 2$

STEP 1: We first show that p (1) is true.

Left Side = $1^2 = 1$

Right Side = $1 (1 + 1) (2*1 + 1)/ 6 = 1$

Both sides of the statement are equal hence p (1) is true.

STEP 2: We now assume that p (k) is true

$1^2 + 2^2 + 3^2 + ... + k^2 = k (k + 1) (2k + 1)/ 6$

and show that p (k + 1) is true by adding $(k + 1)^2$ to both sides of the above statement

$1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 = k (k + 1) (2k + 1)/ 6 + (k + 1)^2$

Set common denominator and factor k + 1 on the right side

$= (k + 1) [k (2k + 1)+ 6 (k + 1)] /6$

Expand k (2k + 1)+ 6 (k + 1)

$= (k + 1) [2k^2 + 7k + 6] /6$

Now factor $2k^2 + 7k + 6$.

$= (k + 1) [(k + 2) (2k + 3)] /6$

We have started from the statement P(k) and have shown that

$1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 = (k + 1) [(k + 2) (2k + 3)] /6$

Which is the statement P(k + 1).

Problem 3

Use mathematical induction to prove that

$\mathbf{1^3 + 2^3 + 3^3 + ... + n^3 = n^2 (n + 1)^2 / 4}$

for all positive integers n.

Solution to Problem 3:

Statement P (n) is defined by

$1^3 + 2^3 + 3^3$

$+ \dots + n^3 = n^2 (n + 1)^2 / 4$

STEP 1: We first show that p (1) is true.

Left Side = $1^3 = 1$

Right Side = $1^2 (1 + 1)^2 / 4 = 1$

hence p (1) is true.

STEP 2: We now assume that p (k) is true

$1^3 + 2^3 + 3^3 + \dots + k^3 = k^2 (k + 1)^2 / 4$

add $(k + 1)^3$ to both sides

$1^3 + 2^3 + 3^3 + \dots + k^3 + (k + 1)^3 = k^2 (k + 1)^2 / 4 + (k + 1)^3$

factor $(k + 1)^2$ on the right side

$= (k + 1)^2 [k^2 / 4 + (k + 1)]$

set to common denominator and group

$= (k + 1)^2 [k^2 + 4 k + 4] / 4$

$= (k + 1)^2 [(k + 2)^2] / 4$

We have started from the statement P(k) and have shown that

$1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3 = (k + 1)^2 [(k + 2)^2] / 4$

Which is the statement P(k + 1).

Problem 4

Prove that for any positive integer number n , $\mathbf{n^3 + 2 n}$ is divisible by 3

Solution to Problem 4:

Statement P (n) is defined by

$n^3 + 2 n$ is divisible by 3

STEP 1: We first show that p (1) is true. Let n = 1 and calculate $n^3 + 2n$

$1^3 + 2(1) = 3$

3 is divisible by 3

hence p (1) is true.

STEP 2: We now assume that p (k) is true

$k^3 + 2k$ is divisible by 3

is equivalent to

$k^3 + 2k = 3M$, where M is a positive integer.

We now consider the algebraic expression $(k + 1)^3 + 2(k + 1)$; expand it and group like terms

$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3$

$= [k^3 + 2k] + [3k^2 + 3k + 3]$

$= 3M + 3[k^2 + k + 1] = 3[M + k^2 + k + 1]$

Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

Problem 5

Prove that $\mathbf{3^n > n^2}$ for n = 1, n = 2 and use the mathematical induction to prove that $3^n > n^2$ for n a positive integer greater than 2.

Solution to Problem 5:

Statement P (n) is defined by

$3^n > n^2$

STEP 1: We first show that p (1) is true. Let n = 1 and calculate 3^1 and 1^2 and compare them

$3^1 = 3$

$1^2 = 1$

3 is greater than 1 and hence p (1) is true.

Let us also show that P(2) is true.

$3^2 = 9$

$2^2 = 4$

Hence P(2) is also true.

STEP 2: We now assume that p (k) is true

$3^k > k^2$

Multiply both sides of the above inequality by 3

$3 * 3^k > 3 * k^2$

The left side is equal to 3^{k+1}. For k >, 2, we can write

$k^2 > 2k$ and $k^2 > 1$

We now combine the above inequalities by adding the left hand sides and the

right hand sides of the two inequalities

$2k^2 > 2k + 1$

We now add k^2 to both sides of the above inequality to obtain the inequality

$3k^2 > k^2 + 2k + 1$

Factor the right side we can write

$3 * k^2 > (k + 1)^2$

If $3 * 3^k > 3 * k^2$ and $3 * k^2 > (k + 1)^{2 \text{ then}}$

$3 * 3^k > (k + 1)^2$

Rewrite the left side as 3^{k+1}

$3^{k+1} > (k + 1)^2$

Which proves tha P(k + 1) is true

Problem 6

Prove that **$n! > 2^n$** for n a positive integer greater than or equal to 4. (Note: n! is n factorial and is given by 1 * 2 * ...* (n-1)*n.)

<u>Solution to Problem 6:</u>

Statement P (n) is defined by

$n! > 2^n$

STEP 1: We first show that p (4) is true. Let n = 4 and calculate 4 ! and 2^n and compare them

$4! = 24$

$2^4 = 16$

24 is greater than 16 and hence p (4) is true.

STEP 2: We now assume that p (k) is true

$k! > 2^k$

Multiply both sides of the above inequality by k + 1

$k! (k + 1) > 2^k (k + 1)$

The left side is equal to (k + 1)!. For k >, 4, we can write

$k + 1 > 2$

Multiply both sides of the above inequality by 2^k to obtain

$2^k (k + 1) > 2 * 2^k$

The above inequality may be written

$2^k (k + 1) > 2^{k + 1}$

We have proved that $(k + 1)! > 2^k (k + 1)$ and $2^k (k + 1) > 2^{k + 1}$ we can now write

$(k + 1)! > 2^{k + 1}$

We have assumed that statement P(k) is true and proved that statment P(k+1) is also true.

Problem 7

Use mathematical induction to prove De Moivre's theorem

$$[R (\cos t + i \sin t)]^n = R^n(\cos nt + i \sin nt)$$

for n a positive integer.

Solution to Problem 7:

STEP 1: For n = 1

$[R (\cos t + i \sin t)]^1 = R^1(\cos 1*t + i \sin 1*t)$

It can easily be seen that the two sides are equal.

STEP 2: We now assume that the theorem is true for n = k, hence

$[R (\cos t + i \sin t)]^k = R^k(\cos kt + i \sin kt)$

Multiply both sides of the above equation by R (cos t + i sin t)

$[R (\cos t + i \sin t)]^k R (\cos t + i \sin t) = R^k(\cos kt + i \sin kt) R (\cos t + i \sin t)$

Rewrite the above as follows

$[R (\cos t + i \sin t)]^{k+1} = R^{k+1} [(\cos kt \cos t - \sin kt \sin t) + i (\sin kt \cos t + \cos kt \sin t)]$

Trigonometric identities can be used to write the trigonometric expressions $(\cos kt \cos t - \sin kt \sin t)$ and $(\sin kt \cos t + \cos kt \sin t)$ as follows

$(\cos kt \cos t - \sin kt \sin t) = \cos(kt + t) = \cos(k + 1)t$

$(\sin kt \cos t + \cos kt \sin t) = \sin(kt + t) = \sin(k + 1)t$

Substitute the above into the last equation to obtain

$[R (\cos t + i \sin t)]^{k+1} = R^{k+1} [\cos (k + 1)t + \sin(k + 1)t]$

It has been established that the theorem is true for n = 1 and that if it assumed true for n = k it is true for n = k + 1.

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